# THE UNIVERSITY
# OF WISCONSIN

AD 787984

MAR 1 1972

*Address.*

Mathematics Research
Center
The University of Wisconsin
Madison. Wisconsin 53706
U.S.A.

N/R-PART I
(MRC-TSR-997)

23

THE UNIVERSITY OF WISCONSIN

MATHEMATICS RESEARCH CENTER

ON POLYNOMIAL SPLINE FUNCTIONS ON THE
CIRCLE II.   MONOSPLINES AND QUADRATURE
FORMULAE

I. J. Schoenberg

D
R
MAR 1 19

Madison, Wisconsin

**DOCUMENT CONTROL DATA - R & D**

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| 1. ORIGINATING ACTIVITY (Corporate author) | 2a. REPORT SECURITY CLASSIFICATION |
|---|---|
| Mathematics Research Center | Unclassified |
| University of Wisconsin, Madison, Wis. 53706 | 2b. GROUP None |

3. REPORT TITLE

ON POLYNOMIAL SPLINE FUNCTIONS ON THE CIRCLE II.   MONOSPLINES AND QUADRATURE FORMULAE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Summary Report: no specific reporting period.

5. AUTHOR(S) (First name, middle initial, last name)

I. J. Schoenberg

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| December 1971 | 19 pp. | 4 |

| 8a. CONTRACT OR GRANT NO. | 9a. ORIGINATOR'S REPORT NUMBER(S) |
|---|---|
| Contract No.   DA-31-124-ARO-D-462 | 1002 |
| 8b. PROJECT NO. None | |
| 8c. | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |
| 8d. | None |

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY |
|---|---|
| None | Army Research Office-Durham, N.C. |

13. ABSTRACT

This is a continuation of MRC Technical Summary Report #997.  The connection between monosplines and quadrature formulae is used to determine the best quadrature formula for $\int f(z)dz$  integrated along the unit circle.  The quadrature formula exact for all spline functions is also derived.

ABSTRACT

This is a continuation of MRC Technical Summary Report #997. The connection between monosplines and quadrature formulae is used to determine the best quadrature formula for $\int f(z)dz$ integrated along the unit circle. The quadrature formula exact for all spline functions is also derived.

# ON POLYNOMIAL SPLINE FUNCTIONS ON THE CIRCLE II.

# MONOSPLINES AND QUADRATURE FORMULAE

I. J. Schoenberg

## Introduction

In [4] we have studied the interpolatory properties of the class $\mathsf{S}_{m,k}$ of polynomial spline functions on the circle that were first discussed by Ahlberg, Nilson and Walsh (see [1], also for further references). In the present second paper we discuss quadrature formulae of the form

$$(1) \qquad \int_U f(z)dz = \sum_0^{k-1} C_j f(\omega^j) + Rf \quad ,$$

where $U$ is the unit circle $|z| = 1$ described counter-clockwise and $\omega = \exp(2\pi i/k)$ . This problem requires a discussion of monosplines and will suggest several classes of spline functions and monosplines having noteworthy properties. Free use is made of the results of [4], in particular of B-splines and their Fourier series expansions.

1. <u>Monosplines and quadrature formulae</u>. Let $k$ and $m$ be positive integers. Our objective is to derive quadrature formulae of the form (1) which are to be exact for polynomials of degree not exceeding $m-1$, i.e.

$$(1.1) \qquad Rf = 0 \quad \text{if} \quad f \in \pi_{m-1} \quad .$$

This property is equivalent to the relations

$$(1.2) \qquad \int_U z^\nu dz = 0 = \sum_{j=0}^{k-1} C_j \omega^{j\nu} \quad , \qquad (\nu = 0,1,\dots,m-1) \quad ,$$

and these show that the quadrature formula (1) has the property (1.1) if and only if

(1.3)    the polynomial $\sum_{0}^{k-1} C_j x^j$ vanishes if $x = 1, \omega, \ldots, \omega^{m-1}$ .

We conclude the following: 1. If $m \geq k$ then the polynomial (1.3) vanishes identically, hence $C_j = 0$ for all $j$ . 2. If $m < k$ then the polynomial (1.3) still depends on $k-m$ arbitrary linear parameters. Avoiding trivial cases we shall assume that

(1.4)                         $1 \leq m < k$  .

Expanding the binomials appearing in (1.5) we see that the condition (1.3) is equivalent to the validity of the identity in $z$

(1.5)                         $\sum_{0}^{k-1} C_j (z - \omega^j)^{m-1} = 0$  .

At this point we reintroduce the class $\mathbf{S}_{m,k}$ of polynomial spline functions on U of degree $m-1$ having the $\omega^j$ as knots, and for a good reason. In fact the identity (1.5) is the precise condition for the existence of a function $S(z) \in \mathbf{S}_{m,k}$ such that

(1.6)    $C_j$ = the jump of $S^{(m-1)}(z)$ at $z = \omega^j$    $(j = 0, \ldots, k-1)$  .

Indeed, if $S(z) = Q(z) \in \pi_{m-1}$ on the arc $(\omega^{k-1}, 1)$ then the function on U defined on the arcs $(\omega^{k-1}, 1), (1, \omega), \ldots, (\omega^{k-1}, 1)$ by the successive partial sums of the expression

$$Q(z) + \sum_{0}^{k-1} C_j \frac{(z - \omega^j)^{m-1}}{(m-1)!}$$

is, in view of (1.5), a spline function $S(z)$ with the property (1.6).

The connection of the spline function $S(z)$ with the quadrature formula (1) is close as shown by

Lemma 1.   **If we write**

$$(1.7) \qquad\qquad K(z) = \frac{1}{m!} z^m - S(z)$$

and if

$$(1.8) \qquad\qquad f(z) \in C^m(U)$$

then we have the relation

$$(1.9) \qquad \int_U f(z)\,dz = \sum_0^{k-1} C_j f(\omega^j) + (-1)^m \int_U K(z) f^{(m)}(z)\,dz \quad .$$

Proof. By (1.6) and (1.7) we find that

$$(1.10) \qquad\qquad -C_j = \text{the jump of } K^{(m-1)}(z) \text{ at } z = \omega^j \quad .$$

Integrations by parts show that

$$\int_U K(z) f^{(m)}(z)\,dz = -\int_U K'(z) f^{(m-1)}(z)\,dz = \ldots = (-1)^{m-1} \int_U K^{(m-1)}(z) f'(z)\,dz$$

a point beyond which we can not continue the process because of the discontinuities of $K^{(m-1)}(z)$ . However, we may proceed as follows

$$\int_U K(z) f^{(m)}(z)\,dz = (-1)^{m-1} \sum_j \int_{\omega^j}^{\omega^{j+1}} K^{(m-1)}(z) f'(z)\,dz$$

$$= (-1)^{m-1} \sum_j \left\{ K^{(m-1)}(\omega_-^{j+1}) - K^{(m-1)}(\omega_+^j) f(\omega^j) - \int_{\omega^j}^{\omega^{j+1}} f(z)\,dK^{(m-1)}(z) \right\}$$

and using (1.10) we obtain

$$\int_U K(z) f^{(m)}(t)\,dz = (-1)^{m-1} \sum C_j f(\omega^j) + (-1)^m \int_U f(z)\,dz$$

because $d K^{(m-1)}(z) = dz$ within each of the arcs.

Using a term familiar from the real axis case (see e.g. [3]) we call $K(z)$ a monospline of degree m and denote their class by the symbol $\mathbb{M}_{m,k}$ . We may then summarize our results as follows: To every quadrature formula (1) with the property (1.1) corresponds a monospline $K(z)$ producing the identity (1.9) . Conversely, to every monospline $K(z)$ corresponds a quadrature

formula (1), the $C_j$ being described by (1.6). This correspondence is one-to-one up to an element of $\pi_{m-1}$ that we may add to $K(z)$ without changing our results. Our discussion of the quadrature formula (1) will single out certain spline functions and monosplines for special study.

2. **The flower-shaped spline functions.** Let $S(z) \in S_{m,k}$ and let us assume that it satisfies the functional equation

(2.1) $$S(z\omega) = \omega^r S(z) , \qquad (|z| = 1) ,$$

where r is an integer, $0 \leq r \leq k-1$. We could say that $S(z)$ is quasiperiodic, or r-quasiperiodic. However, in view of the rotational symmetry of the image of U by $w = S(z)$ we prefer to say that $S(z)$ is flower-shaped and also that $S(z)$ is an r-flower (see Fig. 1 in [4] showing the image of U by a 3-flower for $k = 5$, $m = 2$).

In terms of the B-spline $M_m(z)$ of [4] we have

Lemma 2. For each r, $0 \leq r \leq k-1$, there is up to a constant factor a unique r-flower given by the formula

(2.2) $$S(z) = \sum_{j=0}^{k-1} \omega^{jr} M_m(z\omega^{-j}) .$$

Proof: By [4, Lemma 1] we may write

$$S(z) = \sum c_j M_m(z\omega^{-j}) ,$$

and (2.1) gives $\sum c_j M_m(z\omega^{1-j}) = \sum c_j \omega^r M_m(z\omega^{-j})$ or $\sum c_{j+1} M_m(z\omega^{-j}) = \sum c_j \omega^r M_m(z\omega^{-j})$, whence $c_{j+1} = c_j \omega^r$ and finally $c_j = c_0 \omega^{jr}$. Setting $c_0 = 1$ we obtain (2.2). Conversely, it is easily established that (2.2) satisfies (2.1) and Lemma 2 is established.

The Fourier expansions of r-flowers are described by

**Lemma 3.** *If* $S(z)$ *is an* r-flower then, up to a constant factor,

(2.3) $$S(z) = z^r \quad \text{if} \quad r = 0, 1, \ldots, m-1 \quad ,$$

and

(2.4) $$S(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+r)(ks+r-1)\ldots(ks+r-m+1)} z^{ks+r}$$

$$\underline{\text{if}} \quad r = m, m+1, \ldots, k-1 \quad .$$

Proof: We recall [4, formula (2.16)] that

(2.5) $$M_m(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_\nu b_{\nu-1} \ldots b_{\nu-m+1} z^\nu$$

where

(2.6) $$b_\nu = \frac{1-\omega^{-\nu}}{\nu} \quad \text{if} \quad \nu \neq 0 , \quad b_0 = \frac{2\pi i}{k} \quad .$$

From (2.2) and (2.5)

$$S(z) = \frac{1}{2\pi i} \sum_\nu b_\nu \ldots b_{\nu-m+1} \left( \sum_{j=0}^{k-1} \omega^{jr} \omega^{-j\nu} \right) z^\nu$$

or

(2.7) $$S(z) = \frac{k}{2\pi i} \sum_\nu b_\nu b_{\nu-1} \ldots b_{\nu-m+1} z^\nu \quad ,$$

where the summation is over all $\nu \equiv r \pmod{k}$, hence $\nu = ks+r$. By (2.6) we see that $b_\nu = 0$ if $\nu$ is a nonvanishing multiple of $k$. If $0 \leqq r \leqq m-1$ then (2.7) is seen to reduce to the single term corresponding to $s = 0$ or $\nu = r$ and this is indeed proportional to (2.3). Also directly it is clear that $z^r \epsilon \, \mathcal{S}_{m,k}$ and that $z^r$ is an r-flower.

If $m \leqq r \leqq k-1$, we may discard on the right side of (2.7) the non-vanishing factor

$$\frac{k}{2\pi i} (1-\omega^{-r})(1-\omega^{-r+1}) \ldots (1-\omega^{-r+m-1})$$

and we obtain the expansion (2.4).

#1002

3. **The monospline of least $L_2$-norm.** We have already used the fact

that

$$(3.1) \qquad S(z) = \sum_{0}^{k-1} c_j M_m(z\,\omega^{-j})$$

represents the most general element of $S_{m,k}$. Using (2.5) we find its

Fourier expansion to be

$$S(z) = \frac{1}{2\pi i} \sum_{\nu} b_{\nu} b_{\nu-1} \cdots b_{\nu-m+1} z^{\nu} \sum_{j=0}^{k-1} c_j \omega^{-\nu j} \quad.$$

However

$$(3.2) \qquad \eta_{\nu} = \sum_{j=0}^{k-1} c_j \omega^{-\nu j}$$

in the finite Fourier series representation of an arbitrary periodic sequence

$(\eta_{\nu})$ of period $k$. This establishes

Lemma 4. The most general element of $S_{m,k}$ is given by

$$(3.3) \qquad S(z) = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} b_{\nu} b_{\nu-1} \cdots b_{\nu-m+1} \eta_{\nu} z^{\nu}$$

where $(\eta_{\nu})$ is an arbitrary periodic sequence of numbers of period $k$.

It is now easy to determine the monospline

$$(3.4) \qquad K(z) = \frac{1}{m!} z^m - S(z) , \qquad S(z) \in S_{m,k}$$

having least $L_2$-norm, or

$$(3.5) \qquad \|K\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |K(z)|^2 \, d\theta\right)^{\frac{1}{2}} = \text{minimum} , \qquad (z = e^{i\theta}) \quad.$$

Indeed, by Lemma 4 we may write

$$K(z) = \frac{1}{m!} z^m - \frac{1}{2\pi i} \sum_{\nu} b_{\nu} \cdots b_{\nu-m+1} z^{\nu} \eta_{\nu}$$

$$= \left(\frac{1}{m!} - \frac{1}{2\pi i} b_m \cdots b_1 \eta_m\right) z^m - \frac{1}{2\pi i} \sum_{\nu \neq m} b_{\nu} \cdots b_{\nu-m+1} \eta_{\nu} z^{\nu}$$

and using Parseval's relation we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \left| K(z) \right|^2 d\theta = \left| \frac{1}{m!} - \frac{1}{2\pi i} b_m \cdots b_1 \eta_m \right|^2 + \frac{\left| \eta_m \right|^2}{4\pi^2} \sum_{\substack{\nu \equiv m(k) \\ \nu \neq m}} \left| b_\nu \cdots b_{\nu-m+1} \right|^2$$

(3.6)

$$+ \frac{1}{4\pi^2} \sum_{\substack{\ell=0 \\ \ell \neq m}}^{k-1} \left| \eta_\ell \right|^2 \sum_{\nu \equiv \ell(k)} \left| b_\nu \cdots b_{\nu-m+1} \right|^2 \quad .$$

Observe that by (2.6)

$$\sum_{\nu \equiv \ell(k)} \left| b_\nu \cdots b_{\nu-m+1} \right|^2 > 0 \qquad \text{for all integer } \ell \quad .$$

It follows that in minimizing the right side of (3.6) we must have

(3.7) $\qquad \eta_\ell = 0 \quad$ if $\quad \ell = 0, 1, \ldots, m-1, m+1, \ldots, k-1$

Therefore the search reduces to the much narrower class of monosplines of the form

(3.8) $\qquad K(z) = \frac{1}{m!} z^m - \frac{1}{2\pi i} \eta_m \sum_{\nu \equiv m(k)} b_\nu \cdots b_{\nu-m+1} z^\nu \quad .$

In fact, by Lemma 3, we see that (3.8) describes the most general monospline $K(z)$ satisfying the relation

(3.9) $\qquad\qquad\qquad K(z\omega) = \omega^m K(z) \quad .$

Setting

$$\lambda = \frac{1}{2\pi i} (1-\omega^{-m})(1-\omega^{-m+1}) \cdots (1-\omega^{-1}) \eta_m$$

we may write (3.8) as

(3.10) $\qquad K(z) = \frac{1}{m!} z^m - \lambda \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+z)\cdots(ks+m)} z^{ks+m} \quad ,$

and (3.6) becomes

(3.11) $\quad (m!)^2 (\|K\|_2)^2 = |1-\lambda|^2 + |\lambda|^2 \sum_{s \neq 0} \frac{(m!)^2}{(ks+1)^2(ks+2)^2 \cdots (ks+m)^2} \quad ,$

and we are to determine the value of $\lambda$ that minimizes the right side. The

inequalities $|1-\lambda| \geqq |1-\text{Re}\,\lambda|$ , $|\lambda| \geqq |\text{Re}\,\lambda|$ show that this value of $\lambda$ must be real. Writing

(3.12)
$$c = \sum_{s \neq 0} \frac{(m!)^2}{(ks+1)^2 \ldots (ks+m)^2}$$

we see from the identity

$$(1-\lambda)^2 + c\lambda^2 = (1+c)(\lambda - \frac{1}{1+c})^2 + \frac{c}{1+c} \quad , \qquad (c>0) \quad ,$$

that the least value of the left-hand side is $= c/(1+c)$ and that it is assumed if $\lambda = \lambda_* = 1/(1+c)$ . By (3.12) we see that the right side of (3.11) is least if $\lambda$ equals

(3.13)
$$\lambda_* = \left( \sum_{s=-\infty}^{\infty} \frac{(m!)^2}{(ks+1)^2 (ks+2)^2 \ldots (ks+m)^2} \right)^{-1} \quad .$$

The least value of the right side of (3.11) we now find to be equal $c/(1+c) = (\lambda_*^{-1} - 1)\lambda_* = 1 - \lambda_*$ .

We have therefore established

Theorem 1. The unique monospline (3.4) of least $L_2$-norm is

(3.14)
$$K_*(z) = \frac{1}{m!} z^m - \lambda_* S_m(z)$$

where

(3.15)
$$S_m(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2) \ldots (ks+m)} z^{ks+m} \quad ,$$

while $\lambda_*$ is given by (3.13). The value of the least $L_2$-norm is

(3.16)
$$\|K_*\|_2 = \frac{1}{m!} (1 - \lambda_*)^{\frac{1}{2}} \quad .$$

4. The best quadrature formula (1.9). We obtain the best quadrature formula (1.9), by the definition of the term "best" introduced by A. Sard [2], if we choose in (1.9) $K(z) = K_*(z)$ , and there remains to determine the

coefficients $C_j$ according to (1.10). However, we have already noticed that $K_*(z)$ satisfies the relation (3.9) which by differentiations yields $K_*^{(\nu)}(z\omega) = \omega^{m-\nu}K_*(z)$ and in particular

$$K_*^{(m-1)}(z\omega) = \omega K_*^{(m-1)}(z) \quad .$$

By (1.10) we see that

$$(4.1) \qquad\qquad C_j = \omega^j C_0$$

and it therefore suffices to determine the jump $\Delta_0 = -C_0$ of $K^{(m-1)}(z)$ at $z = 1$ . It is easily obtained as follows.

Differentiating the Fourier series (3.14) $m-1$ times we get

$$(4.2) \qquad\qquad K_*^{(m-1)}(z) = z - \lambda_* \sum_s \frac{1}{ks+1} z^{ks+1} \quad .$$

Its sum is, as we know, a step function with discontinuities at the $\omega^j$ . To determine its jump $\Delta_0$ at $z = 1$ , we consider the function $\varphi(\theta)$ of period $2\pi$ such that

$$(4.3) \qquad\qquad \varphi(\theta) = \tfrac{1}{2}(\pi-\theta) \quad \text{if} \quad 0 < \theta < 2\pi \quad .$$

Defining also $\varphi(0) = 0$ , we have the familiar expansion

$$(4.4) \qquad \varphi(\theta) = \sum_1^\infty \frac{\sin\nu\theta}{\nu} = \frac{1}{2i} \sum_{s\neq 0} \frac{1}{s} e^{is\theta} \quad \text{for all real } \theta ,$$

from which we derive that

$$\frac{1}{k}\varphi(k\theta)e^{i\theta} = \frac{1}{2i} \sum_{s\neq 0} \frac{1}{ks} z^{ks+1} , \qquad (z = e^{i\theta}) \quad .$$

Using (4.2) we find the expansion

$$(4.5) \quad K_*^{(m-1)}(e^{i\theta}) + 2i\lambda_* \frac{1}{k}\varphi(k\theta)e^{i\theta} = (1-\lambda_*)z + \lambda_* \sum_{s\neq 0} \frac{1}{ks(ks+1)} z^{ks+1}$$

which is seen to converge absolutely and uniformly on $U$ . We conclude that

the left side belongs to $C(U)$. The jump of the function $\varphi(\theta)$ being $= \pi$, at $\theta = 0$, by (4.3), we conclude that the jump $2i\lambda_* k^{-1}\pi$ of the second term of the left side of (4.5) cancels the jump $\Delta_0$ of the first term. Therefore

$$(4.6) \qquad \Delta_0 = -\frac{2\pi i}{k}\lambda_* \ .$$

Using the relations (4.1) we obtain

Theorem 2. Let $1 \leqq m < k$ . Among all quadrature formulae of the form (1) and having the property (1.1), the best is the quadrature formula

$$(4.7) \qquad \int_z f(z)dz = \lambda_* 2\pi i k^{-1}\sum_{j=0}^{k-1} \omega^j f(\omega^j) + (-1)^m \int_U K_*(z)f^{(m)}(z)dz$$

where $K_*(z)$ is the monospline of Theorem 1, while the positive fraction $\lambda_*$ is defined by (3.13) .

Theorem 2 does not state or imply that the familiar average (or Riemann sum)

$$(4.8) \qquad A = 2\pi i k^{-1}\sum_0^{k-1} \omega^j f(\omega^j)$$

is not as good an approximation of the integral as the modified approximation $\lambda_* A$ appearing in (4.7) . However, Theorem 2 does state that $\lambda_* A$ is the best approximation in the following sense: For all functions $f(z)$ such that

$$(4.9) \qquad \int_0^{2\pi} |f^{(m)}(z)|^2 d\theta \leqq H^2 \ ,$$

where $H$ is a fixed constant, we have the inequality

$$(4.10) \qquad |\int_U f(z)dz - \lambda_* A| \leqq J_* \cdot H \ , \qquad J_* = \frac{1}{m!}(2\pi(1-\lambda_*))^{\frac{1}{2}} \ .$$

For any other inequality

$$|\int_U f(z)dz - \sum_0^{k-1} C_j f(\omega^j)| \leqq J \cdot H$$

valid for the class (4.9), we must have $J > J_*$ .

5. **The quadrature formula that is exact if** $f \in \mathcal{S}_{m,k}$ . We wish to determine the quadrature formula

(5.1)
$$\int_U f(z)dz = \sum_0^{k-1} \widetilde{C}_j f(\omega^j) + \widetilde{R}f$$

such that

(5.2)
$$\widetilde{R}f = 0 \quad \text{if} \quad f \in \mathcal{S}_{m,k} \quad .$$

One way to construct such a formula is as follows. We assume that the class $\mathcal{S}_{m,k}$ has the interpolatory property at the knots $\omega^j$ . By [4, Theorem 1] we must

(5.3) **exclude the case that** $k$ **is even and** $m$ **is odd.**

We are then assured [4] of the existence of a fundamental function $L_m(z) \in \mathcal{S}_{m,k}$ such that

(5.4)
$$f(z) = \sum_0^{k-1} f(\omega^j)L_m(z\omega^{-j}) \quad \text{if} \quad f \in \mathcal{S}_{m,k} \quad .$$

Integrating both sides of this identity along $U$ we see that (5.1) has the property (5.2) if we set

(5.5)
$$\widetilde{C}_j = \int_U L_m(z\omega^{-j})dz \quad .$$

Again we assume (5.3). A second way of constructing this quadrature formula is as follows. Let $S_0(t)$ be the unique spline function such that the monospline

(5.6)
$$K_0(z) = \frac{1}{m!} z^m - S_0(z)$$

has the property

(5.7)
$$K_0(\omega^j) = 0 \qquad (j = 0, \ldots, k-1) \quad .$$

The existence and unicity of $S_0(z)$ follow from [4, Theorem 1]. We shall now establish several properties of $K_0(z)$ which will show that this monospline will produce the quadrature formula we are looking for.

I. **The monospline** (5.6) **satisfies**

(5.8)
$$K_0(z\omega) = \omega^m K_0(z) \quad .$$

**Proof:** Clearly

$$\omega^{-m} K_0(z\omega) = \frac{1}{m!} z^m - \omega^{-m} S_0(z\omega)$$

is a monospline satisfying the relations (5.7). The unicity of $K_0(z)$ implies the relation (5.8).

II. **The quadrature formula**

(5.9)
$$\int_U f(z)dz = \sum_0^{k-1} C_j^0 f(\omega^j) + R_0 f \quad ,$$

**where** $- C_j^0 =$ **the jump of** $K_0^{(m-1)}(z)$ **at** $z = \omega^j$ , **is exact if** $f \in \mathbf{S}_{m,k}$ .

**Proof:** By Lemma 1 we may write

$$\pm R_0 f = \int_U K_0(z)f^{(m)}(z)dz = \int_U K_0(z)df^{(m-1)}(z) \quad \text{if} \quad f \in C^m(U) \quad .$$

However, the last form

(5.10)
$$\pm R_0 f = \int_U K_0(z)df^{(m-1)}(z)$$

is also applicable (see proof of Lemma 1) if we only assume $f^{(m-2)}(z)$ to be absolutely continuous and $f^{(m-1)}(z)$ of bounded variation on $U$ . We may therefore assume in (5.9), (5.10) that $f(z) \in \mathbf{S}_{m,k}$ and, therefore, that $f^{(m-1)}(z)$ is a step function. If we denote by $\delta_j$ its jump at $z = \omega^j$ , then (5.7) and (5.10) show that

$$\pm R_0 f = \sum K_0(\omega^j)\delta_j = 0$$

III. $\qquad C_j^0 = \widetilde{C}_j \qquad (j = 0, \ldots, k-1)$ .

**Proof:** Letting $f(z) = L_m(z\omega^{-j})$ in (5.9), then (5.5) shows that

$$C_j^0 = \int_U L_m(z\omega^{-j})dz = \widetilde{C}_j \quad .$$

The explicit construction of the quadrature formula (5.9) now presents

no difficulties. We conclude from the relation (5.8) that $S_0(z)$ in an

m-flower and Lemma 3 implies that $K_0(z)$ may be written as

$$(5.11) \qquad K_0(z) = \frac{1}{m!} z^m - \lambda_0 \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2)\ldots(ks+m)} z^{ks+m}$$

for an appropriate value of the constant $\lambda_0$ . The defining properties (5.7),

in particular $K_0(1) = 0$ show that

$$(5.12) \qquad \lambda_0 = \left( \sum_{s=-\infty}^{\infty} \frac{m!}{(ks+1)(ks+2)\ldots(ks+m)} \right)^{-1} \quad .$$

This requires that the sum of the series in (5.12) should not vanish. By

[4, Lemma 4] this series can vanish only if $m = k-1$ and $m$ is odd. But

this would imply that $k$ is even and $m$ is odd, a situation which can not occur

because of our assumption (5.3) .

We summarize our results in

Theorem 3. <u>If we exclude the case that</u>

$$(5.13) \qquad m = k-1 \quad \underline{and} \quad m \quad \underline{is\ odd} \ ,$$

then

$$(5.14) \qquad \int_U f(z)dz = \lambda_0 2\pi i k^{-1} \sum_{0}^{k-1} \omega^j f(\omega^j) + (-1)^m \int_U K_0(z) f^{(m)}(z)dz$$

<u>is a quadrature formula that is exact if</u> $f(z) \in \mathbf{8}_{m,k}$ , <u>where</u> $K_0(z)$ <u>and</u> $\lambda_0$

<u>are described by</u> (5.11) <u>and</u> (5.12) . <u>If we replace</u> (5.13) <u>by the more stringent</u>

<u>assumption</u> (5.3) <u>then the quadrature formula</u> (5.14) <u>may also be obtained by</u>

<u>integration from the spline interpolation formula</u> (5.4) .

The value $C_0 = 2\pi i \lambda_0 / k$ used in writing the formula (5.14) follow from (4.6) on replacing $\lambda_*$ by $\lambda_0$ .

6. <u>The polynomial components of a certain spline function.</u>  The spline function

$$(6.1) \qquad S_m(z) = \sum_{s=-\infty}^{\infty} \frac{1}{(ks+1)(ks+2)\ldots(ks+m)} z^{ks+m}$$

appears in the representation (3.14), (3.15) of $K_*(z)$ as well as in (5.1!) which describes $K_0(z)$ .  By Lemma 3 it may also be characterized as an m-flower of degree m-1 .  Here we propose to determine explicitly its individual polynomial components.  Without loss of generality we may assume that $m = k-1$ and determine the components of

$$(6.2) \qquad S_{k-1}(z) = \sum_{-\infty}^{\infty} \frac{1}{(ks+1)\ldots(ks+k-1)} z^{ks+k-1}$$

it being clear by differentiations that

$$(6.3) \qquad S_m(z) = S_{k-1}^{(k-m-1)}(z) , \qquad (1 \leqq m \leqq k-1) \quad .$$

The functional equation

$$(6.4) \qquad S_{k-1}(z\omega) = \omega^{-1} S_{k-1}(z)$$

shows that it suffices to determine the polynomial component in the arc $(1, \omega)$ .

Setting

$$(6.5) \qquad \psi = \exp(\pi i/k)$$

we turn $S_{k-1}(z)$ by $\psi^{-1}$ to obtain, since $\psi^k = -1$ , the new spline function

$$(6.6) \qquad s(z) = \psi S(z\psi) = \sum_{-\infty}^{\infty} \frac{(-1)^{s+1}}{(ks+1)\ldots(ks+k-1)} z^{ks+k-1}$$

having its knots in the new locations $\psi \omega^j$ .  Let

$$(6.7) \qquad P_0(z) = \alpha_0 \frac{z^{k-2}}{(k-2)!} + \alpha_1 \frac{z^{k-3}}{(k-3)!} + \cdots + \alpha_{k-2}$$

be the component of $s(z)$ in the arc $(\psi^{-1}, \psi)$ . The Laurent series (6.6) having real coefficients it is clear that $\overline{P(z)} = P(\bar{z})$ if $z$ is on the arc $(\psi^{-1}, \psi)$ and therefore also for all $z$ . This shows that $P_0(z)$ is a <u>real</u> polynomial. Also $s(z)$ satisfies the relation $s(z\omega) = \omega^{-1} s(z)$ whence

$$(6.8) \qquad s(z) = \omega^{-1} s(z\omega^{-1}) \quad .$$

If $z$ is in the arc $(\psi, \psi\omega)$, then (6.8) and (6.7) show that the component of $s(z)$ on this arc is

$$(6.9) \qquad P_1(z) = \alpha_0 \omega \frac{z^{k-2}}{(k-2)!} + \alpha_1 \omega^2 \frac{z^{k-3}}{(k-3)!} + \cdots + \alpha_{k-2} \omega^{k-1} \quad .$$

At this point we use that $s(z) \in C^{k-3}(U)$ and obtain at $z = \psi$ the $k-2$ equations

$$(6.10) \qquad P_0^{(\nu)}(\psi) = P_1^{(\nu)}(\psi) \qquad (\nu = 0, \ldots, k-3) \quad .$$

Using (6.7) and (6.9) we obtain the system

$$\alpha_0(\omega-1) \frac{\psi^\nu}{\nu!} + \alpha_1(\omega^2-1) \frac{\psi^{\nu-1}}{(\nu-1)!} + \cdots + \alpha_\nu(\omega^{\nu+1}-1) = 0 \qquad (\nu = 1, \ldots, k-2) \quad .$$

Dividing by $\omega^{\nu+1} - 1$ and using the relations

$$\frac{\omega^\alpha - 1}{\omega^\beta - 1} \cdot \frac{\psi^{\beta-\alpha}}{(\beta-\alpha)!} = \frac{1}{(\beta-\alpha)!} \frac{\psi^\alpha - \psi^{-\alpha}}{\psi^\beta - \psi^{-\beta}} = \frac{1}{(\beta-\alpha)!} \frac{\sin(\pi\alpha/k)}{\sin(\pi\beta/k)}$$

we obtain the system

$$0 = \frac{1}{1!} \alpha_0 \sin(\pi/k) + \alpha_1 \sin(2\pi/k)$$

$$0 = \frac{1}{2!} \alpha_0 \sin(\pi/k) + \frac{1}{1!} \alpha_1 \sin(2\pi/k) + \alpha_2 \sin(3\pi/k)$$

$$\cdot$$
$$\cdot$$

$$0 = \frac{1}{(k-2)!} \alpha_0 \sin(\pi/k) + \frac{1}{(k-3)!} \alpha_1 \sin(2\pi/k) + \cdots + \alpha_{k-2} \sin((k-1)\pi/k)$$

which in terms of the new unknowns

(6.11) $$\beta_\nu = \alpha_\nu \sin \frac{(\nu+1)\pi}{k} \qquad (\nu = 0, \ldots, k-2)$$

becomes

$$0 = \frac{1}{1!} \beta_0 + \beta_1$$

$$0 = \frac{1}{2!} \beta_0 + \frac{1}{1!} \beta_1 + \beta_2$$

$$\vdots$$

$$0 = \frac{1}{(k-2)!} \beta_0 + \frac{1}{(k-3)!} \beta_1 + \cdots + \frac{1}{1!} \beta_{k-3} + \beta_{k-2} \quad .$$

This being the system giving the coefficients of the power series reciprocal

to $e^x = \sum x^\nu/\nu!$ we obtain a solution $\beta_\nu = (-1)^\nu/\nu!$ and by (6.11)

(6.12) $$\alpha_\nu = (-1)^\nu \Big/ \left( \nu! \sin \frac{(\nu+1)\pi}{k} \right), \qquad (\nu = 0, \ldots, k-2) \quad .$$

From (6.6) and (6.7) we therefore obtain the identity

(6.13) $$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\ldots(ks+k-1)} z^{ks+k-1} = c \cdot \sum_{\nu=0}^{k-2} (-1)^\nu \binom{k-2}{\nu} z^{k-\nu-2} \Big/ \sin \frac{(\nu+1)\pi}{k}$$

valid in the arc $(\psi^{-1}, \psi)$, while $c$ is a constant yet to be determined.

To obtain $c$ we set $z = 1$ to obtain

(6.14) $$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\ldots(ks+k-1)} = c \cdot \sum_{0}^{k-2} (-1)^\nu \binom{k-2}{\nu} \Big/ \sin \frac{(\nu+1)\pi}{k}$$

However, the left side is easily evaluated directly from the partial fraction

decomposition

(6.15) $$\frac{1}{(ks+1)\ldots(ks+k-1)} = \frac{1}{(k-2)!} \left\{ \frac{1}{(ks+1)} - \binom{k-2}{1} \frac{1}{ks+2} + \ldots + (-1)^{k-2} \frac{1}{ks+k-1} \right\}$$

From the classical expansion $\sum (-1)^s/(s+x) = \pi/\sin \pi x$ we obtain

$$\sum \frac{(-1)^s}{ks+\nu+1} = \frac{\pi}{k \sin \frac{(\nu+1)\pi}{k}}$$

If we multiply (6.15) by $(-1)^s$ and sum over all $s$ we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^s}{(ks+1)\dots(ks+k-1)} = \frac{\pi}{(k-2)!k} \sum_{0}^{k-2} (-1)^{\nu} \binom{k-2}{\nu} \Big/ \sin \frac{(\nu+1)\pi}{k} \quad .$$

Comparing with (6.14) we find that

$$(6.16) \qquad\qquad c = \frac{\pi}{(k-2)!k} \quad .$$

Using the relation (6.3) we can also determine the polynomial component of (6.1) within the arc $(1,\omega)$ . Indeed

$$\psi^{-m} S_m(z\psi) = \sum_{s} \frac{(-1)^s}{(ks+1)\dots(ks+m)} z^{ks+m}$$

and differentiating (6.13) $k-m-1$ times we obtain that

$$(6.17) \qquad \psi^{-m} S_m(z\psi) = \frac{\pi}{k(m-1)!} \sum_{\nu=0}^{m-1} (-1)^{\nu} \binom{m-1}{\nu} z^{m-\nu-1} \Big/ \sin \frac{(\nu+1)\pi}{k}$$

if $z$ is confined to the arc $(\psi^{-1}, \psi)$ .

As an example let us determine $K_*(z)$ using the formulae (3.14), (3.15) for $k = 5$ and $m = 2$ . The rapidly convergent series (3.13) shows that $\lambda_* = .969690$ , and (6.17) furnishes the expression

$$\psi^{-2} K_*(z\psi) = \tfrac{1}{2} z^2 - \lambda_* \frac{\pi}{5\sin(\pi/5)} \left( z - \frac{1}{2\cos\frac{\pi}{5}} \right)$$

in the arc $(\psi^{-1}, \psi)$ . Graphing the image of the arc $(\psi^{-1}, \psi)$ by the quadratic polynomial on the right side and turning it by $\psi^2 = \omega$ (or $+72^{\circ}$) we obtain the image of the arc $(1,\omega)$ by $w = K_*(z)$ . The functional equation

$$K_*(z\omega) = \omega^2 K_*(z)$$

allows to complete the image of the entire circle $U$ which is shown in Fig. 1 . The five corners of the curve are the images of the $\omega^j$ , Fig. 1 showing in parentheses the corresponding $\omega^j$ . Moreover

$$|K(\omega^j)| = .14063 , \qquad |K(\psi\omega^j)| = .10409 .$$

A general discussion and determination of the monosplines $K(z)$ of least $L_\infty$ - norm would be of interest.
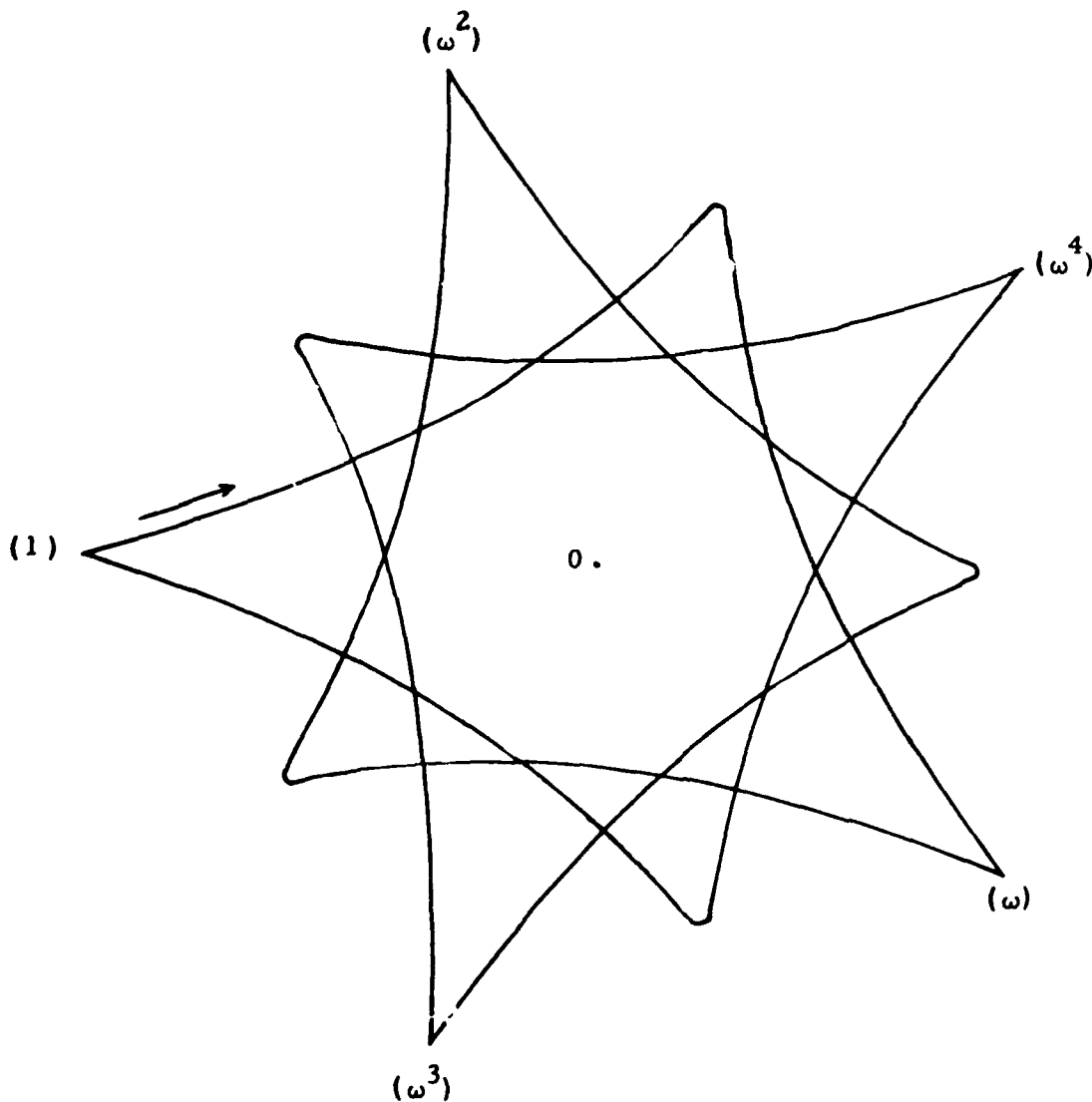


Fig. 1

# REFERENCES

1. J. H. Ahlberg, E. N. Nilson and J. L Walsh, <u>Properties of analytic splines</u> I. <u>Complex polynomial splines,</u> J. Math. Anal. and Applic., <u>27</u> (1969), 262-278.

2. Arth_ar Sard, <u>Linear Approximation,</u> American Mathematical Society, Providence, R. I., 1963.

3. I. J. Schoenberg, <u>Monosplines and quadrature formulae,</u> 157-207 in <u>Theory and Applications of Spline Functions,</u> edited by T. N. E. Greville, Academic Press, New /ork-London, 1969.

4. _____, <u>On polynomial spline functions on the circle</u> I. <u>Their interpolatory properties</u>, MRC Technical Summary Report #997, Madison, Wisconsin.